Higher order Painlevé system of type $D_{2n+2}^{(1)}$ arising from integrable hierarchy

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Abstract

A higher order Painlevé system of type $D_{2n+2}^{(1)}$ was introduced by Y. Sasano. It is an extension of the sixth Painlevé equation (P_{VI}) for the affine Weyl group symmetry. It is also expressed as a Hamiltonian system of order $2n$ with a coupled Hamiltonian of P_{VI}. In this paper, we discuss a derivation of this system from a Drinfeld-Sokolov hierarchy.

1 Introduction

The Drinfeld-Sokolov hierarchies are extensions of the KdV (or mKdV) hierarchy for the affine Lie algebras [DS]. It is known that they imply several Painlevé equations by similarity reduction [AS, FS, KK1, KIK, KK2]. On the other hand, two types of extensions of the Painlevé equations for the affine Weyl group symmetry have been studied, type $A_n^{(1)}$ [NY1] and type $D_{2n+2}^{(1)}$ [S]. For type $A_n^{(1)}$ among them, the relation to the Drinfeld-Sokolov hierarchies is already clarified. In this paper, we investigate the relation for type $D_{2n+2}^{(1)}$.

Recall that the higher order Painlevé system of type $D_{2n+2}^{(1)}$ given in [S] is a Hamiltonian system of order $2n$ with a coupled Hamiltonian of P_{VI}. Let q_i, p_i ($i = 1, \ldots, n$) be dependent variables on s and α_i ($i = 0, \ldots, 2n+2$) complex parameters satisfying

$$\alpha_0 + \alpha_1 + \sum_{j=2}^{2n} 2\alpha_j + \alpha_{2n+1} + \alpha_{2n+2} = 1.$$

We also set

$$\begin{aligned}H_i = q_i(q_i-1)(q_i-s)p_i^2 - \{(\beta_{i,1}-1)q_i(q_i-1) \\ + \beta_{i,3}(q_i-1)(q_i-s) + \beta_{i,4}q_i(q_i-s)\}p_i + \alpha_{2i}(\alpha_{2i}+\beta_{i,0})q_i,\end{aligned}$$

for $i=1,\dots,n$, where

$$\beta_{i,0} = \alpha_1 + \sum_{j=1}^{i-1}\alpha_{2j+1}, \quad \beta_{i,1} = \alpha_0 + \sum_{j=1}^{i-1}2\alpha_{2j} + \sum_{j=1}^{i-1}\alpha_{2j+1},$$
$$\beta_{i,3} = \sum_{j=i}^{n-1}\alpha_{2j+1} + \sum_{j=i+1}^{n}2\alpha_{2j} + \alpha_{2n+1}, \quad \beta_{i,4} = \sum_{j=i}^{n-1}\alpha_{2j+1} + \alpha_{2n+2}.$$

We consider a Hamiltonian system

$$s(s-1)\frac{dq_i}{ds} = \{H, q_i\}, \quad s(s-1)\frac{dp_i}{ds} = \{H, p_i\} \quad (i=1,\dots,n), \tag{1.1}$$

with a Hamiltonian

$$H = \sum_{i=1}^{n} H_i + \sum_{1\le i<j\le n} 2(q_i-s)p_iq_j\{(q_j-1)p_j+\alpha_{2j}\}, \tag{1.2}$$

where $\{\cdot,\cdot\}$ stands for the Poisson bracket defined by

$$\{p_i, q_j\} = \delta_{i,j}, \quad \{p_i,p_j\} = \{q_i,q_j\} = 0 \quad (i,j=1,\dots,n).$$

Note that each H_i is equivalent to the Hamiltonian of P_{VI} (see [IKSY]). In fact, the parameters satisfy the following relations:

$$\beta_{i,0} + \beta_{i,1} + 2\alpha_{2i} + \beta_{i,3} + \beta_{i,4} = 1 \quad (i=1,\dots,n).$$

The system (1.1) with (1.2) admits affine Weyl group symmetry of type $D^{(1)}_{2n+2}$. Denoting the dependent variables by

$$\varphi_0 = \frac{1}{2n+2}, \quad \varphi_1 = q_1 - s, \quad \varphi_{2i+1} = q_{i+1} - q_i \quad (i=1,\dots,n-1),$$
$$\varphi_{2j} = -\frac{p_j}{2n+2} \quad (j=1,\dots,n), \quad \varphi_{2n+1} = 1-q_n, \quad \varphi_{2n+2} = -q_n,$$

we consider birational canonical transformations

$$r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i, \quad r_i(\varphi_j) = \varphi_j + \frac{\alpha_i}{\varphi_i}\{\varphi_i, \varphi_j\}, \tag{1.3}$$

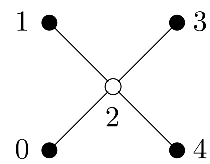

Figure 1: Gradation of $\mathfrak{g}(D_4^{(1)})$ of type $(1,1,0,1,1)$

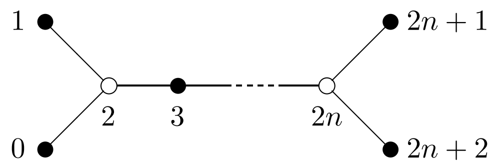

Figure 2: Gradation of $\mathfrak{g}(D_{2n+2}^{(1)})$ of type $(1,1,0,1,0,\dots,1,0,1,1)$

for $i,j=0,\dots,2n+2$, where

$$\begin{aligned} & a_{ii}=2 && (i=0,\dots,2n+2), \\ & a_{02}=a_{ii+1}=a_{2n2n+2}=-1 && (i=1,\dots,2n), \\ & a_{ij}=0 && (\text{otherwise}). \end{aligned}$$

Then the system (1.1) with (1.2) is invariant under the action of them. Furthermore, a group of symmetries $\langle r_0,\dots,r_{2n+2}\rangle$ is isomorphic to the affine Weyl group of type $D_{2n+2}^{(1)}$.

In this paper, we show that the system (1.1) with (1.2) is derived from a Drinfeld-Sokolov hierarchy by similarity reduction. The Drinfeld-Sokolov hierarchies are characterized by graded Heisenberg subalgebras of the affine Lie algebras. For a derivation of (1.1), we choose the affine Lie algebra $\mathfrak{g}(D_{2n+2}^{(1)})$ and its graded Heisenberg subalgebra of type $(1,1,0,1,0,\dots,1,0,1,1)$. It is suggested by the fact that $P_{\rm VI}$ is derived from the hierarchy associated with the graded Heisenberg subalgebra of $\mathfrak{g}(D_4^{(1)})$ of type $(1,1,0,1,1)$.

This paper is organaized as follows. In Section 2, we recall the affine Lie algebra $\mathfrak{g}(D_{2n+2}^{(1)})$ and its graded Heisenberg subalgebra. In Section 3, we formulate a similarity reduction of a Drinfeld-Sokolov hierarchy of type $D_{2n+2}^{(1)}$. In Section 4, we derive the system (1.1) with (1.2) from the similarity reduction. In Section 5, we discuss a derivation of the group of symmetries (1.3).

2 Affine Lie algebra

In this section, we introduce the affine Lie algebra of type $D^{(1)}_{2n+2}$ and its Heisenberg subalgebra of type $(1,1,0,1,0,\dots,1,0,1,1)$, following the notation of [Kac].

Recall that $\mathfrak{g} = \mathfrak{g}(D^{(1)}_{2n+2})$ is a Lie algebra generated by the Chevalley generators e_i, f_i, $\alpha_i^\vee$ $(i = 0,\dots,2n+2)$ and the scaling element d with the fundamental relations

$$
\begin{aligned}
&(\mathrm{ad} e_i)^{1-a_{ij}}(e_j) = 0, \quad (\mathrm{ad} f_i)^{1-a_{ij}}(f_j) = 0 \quad (i \neq j),\\
&[\alpha_i^\vee, \alpha_j^\vee] = 0, \quad [\alpha_i^\vee, e_j] = a_{ij} e_j, \quad [\alpha_i^\vee, f_j] = -a_{ij} f_j, \quad [e_i, f_j] = \delta_{i,j}\alpha_i^\vee,\\
&[d, \alpha_i^\vee] = 0, \quad [d, e_i] = \delta_{i,0} e_0, \quad [d, f_i] = -\delta_{i,0} f_0,
\end{aligned}
$$

for $i, j = 0,\dots,2n+2$. The generalized Cartan matrix $A = (a_{ij})_{i,j=0}^{2n+2}$ for $\mathfrak{g}$ is defined by

$$
\begin{aligned}
&a_{ii} = 2 && (i = 0,\dots,2n+2),\\
&a_{02} = a_{ii+1} = a_{2n2n+2} = -1 && (i = 1,\dots,2n),\\
&a_{ij} = 0 && (\text{otherwise}).
\end{aligned}
$$

We denote the Cartan subalgebra of $\mathfrak{g}$ by

$$
\mathfrak{h} = \bigoplus_{j=0}^{2n+2} \mathbb{C}\alpha_j^\vee \oplus \mathbb{C}d.
$$

The canonical central element of $\mathfrak{g}$ is given by

$$
K = \alpha_0^\vee + \alpha_1^\vee + \sum_{i=2}^{2n} 2\alpha_i^\vee + \alpha_{2n+1}^\vee + \alpha_{2n+2}^\vee.
$$

The normalized invariant form $(\cdot|\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is determined by the conditions

$$
\begin{aligned}
&(\alpha_i^\vee|\alpha_j^\vee) = a_{ij}, \quad (e_i|f_j) = \delta_{i,j}, \quad (\alpha_i^\vee|e_j) = (\alpha_i^\vee|f_j) = 0,\\
&(d|d) = 0, \qquad (d|\alpha_j^\vee) = \delta_{0,j}, \quad (d|e_j) = (d|f_j) = 0,
\end{aligned}
$$

for $i, j = 0,\dots,2n+2$.

Consider a gradation $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$ of type $(1,1,0,1,0,\dots,1,0,1,1)$ by setting

$$
\begin{aligned}
&\deg \mathfrak{h} = \deg e_i = \deg f_i = 0 \quad (i \in \mathcal{I}),\\
&\deg e_j = 1, \quad \deg f_j = -1 \quad (j \in \mathcal{J}),
\end{aligned}
$$

where $\mathcal{I} = \{2, 4, \ldots, 2n\}$ and $\mathcal{J} = \{0, 1, 3, 5, \ldots, 2n+1, 2n+2\}$. With an element $\vartheta \in \mathfrak{h}$ such that

$$(\vartheta|\alpha_i^\vee) = 0, \quad (\vartheta|\alpha_j^\vee) = 1 \quad (i \in \mathcal{I}; j \in \mathcal{J}),$$

this gradation is defined by

$$\mathfrak{g}_k = \left\{ x \in \mathfrak{g} \;\middle|\; [\vartheta, x] = kx \right\} \quad (k \in \mathbb{Z}).$$

We denote by

$$\mathfrak{g}_{<0} = \bigoplus_{k<0} \mathfrak{g}_k, \quad \mathfrak{g}_{\geq 0} = \bigoplus_{k \geq 0} \mathfrak{g}_k.$$

Such gradation implies the Heisenberg subalgebra of $\mathfrak{g}$

$$\mathfrak{s} = \{ x \in \mathfrak{g} \;|\; [x, \Lambda_1] = [x, \Lambda_2] = \mathbb{C}K \},$$

with elements of $\mathfrak{g}_1$

$$\Lambda_1 = e_0 + e_{1,2} + \sum_{j \in \mathcal{J}'} (e_j + e_{j-1,j,j+1}) + e_{2n+1} + e_{2n,2n+2},$$
$$\Lambda_2 = e_1 + e_{0,2} + \sum_{j \in \mathcal{J}'} (e_{j-1,j} + e_{j,j+1}) + e_{2n+2} + e_{2n,2n+1},$$

where $\mathcal{J}' = \{3, 5, \ldots, 2n-1\}$ and

$$e_{i_1,i_2,\ldots,i_{n-1},i_n} = \mathrm{ad}e_{i_1}\mathrm{ad}e_{i_2} \ldots \mathrm{ad}e_{i_{n-1}}(e_{i_n}).$$

Note that $\mathfrak{s}$ admits the gradation of type $(1, 1, 0, 1, 0, \ldots, 1, 0, 1, 1)$, namely

$$\mathfrak{s} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{s}_k, \quad \mathfrak{s}_k \subset \mathfrak{g}_k.$$

We also remark that the positive part of $\mathfrak{s}$ has a graded bases $\{\Lambda_k\}_{k=1}^{\infty}$ satisfying

$$[\Lambda_k, \Lambda_l] = 0, \quad [\vartheta, \Lambda_k] = n_k \Lambda_k \quad (k, l = 1, 2, \ldots),$$

where n_k stands for the degree of element Λ_k defined by

$$n_k = \left\{ \begin{array}{ll} k & (k : \text{odd}) \\ k-1 & (k : \text{even}) \end{array} \right. .$$

The explicit formulas of Λ_k ($k \geq 3$) are given in Appendix A.

In the last, we introduce the Borel subalgebra of $\mathfrak{g}$. Let $\mathfrak{n}_+$ and $\mathfrak{n}_-$ be the subalgebras of $\mathfrak{g}$ generated by e_i and f_i ($i = 0, \ldots, 2n+2$) respectively.

Then the Borel subalgebra $\mathfrak{b}_+$ of $\mathfrak{g}$ is defined by $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$. Note that we have the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ = \mathfrak{n}_- \oplus \mathfrak{b}_+.$$

We also remark that

$$\mathfrak{n}_- = \mathfrak{g}_{<0} \oplus \bigoplus_{i\in\mathcal{I}} \mathbb{C}f_i, \quad \mathfrak{g}_{\geq 0} = \bigoplus_{i\in\mathcal{I}} \mathbb{C}f_i \oplus \mathfrak{b}_+.$$

3 Drinfeld-Sokolov hierarchy

In this section, we formulate a Drinfeld-Sokolov hierarchy of type $D^{(1)}_{2n+2}$ and its similarity reduction associated with the Heisenberg subalgebra $\mathfrak{s}$.

In the following, we use the notation of infinite dimensional groups

$$G_{<0} = \exp(\widehat{\mathfrak{g}}_{<0}), \quad G_{\geq 0} = \exp(\widehat{\mathfrak{g}}_{\geq 0}),$$

where $\widehat{\mathfrak{g}}_{<0}$ and $\widehat{\mathfrak{g}}_{\geq 0}$ are completions of $\mathfrak{g}_{<0}$ and $\mathfrak{g}_{\geq 0}$ respectively.

Introducing the time variables t_k $(k = 1, 2, \ldots)$, we consider a system of partial differential equations

$$\partial_{t_k} - B_k = W(\partial_{t_k} - \Lambda_k)W^{-1} \quad (k = 1, 2, \ldots), \tag{3.1}$$

for a $G_{<0}$-valued function W, where B_k stands for the $\mathfrak{g}_{\geq 0}$-component of $W\Lambda_k W^{-1}$. The Zakharov-Shabat equations

$$[\partial_{t_k} - B_k, \partial_{t_l} - B_l] = 0 \quad (k, l = 1, 2, \ldots), \tag{3.2}$$

follows from the system (3.1). We call the system (3.2) the Drinfeld-Sokolov hierarchy of type $D^{(1)}_{2n+2}$.

Under the system (3.1), we consider the operator

$$\mathcal{M} = W \exp\left(\sum_{k=1,2,\ldots} t_k\Lambda_k\right) \vartheta \exp\left(-\sum_{k=1,2,\ldots} t_k\Lambda_k\right) W^{-1}.$$

Then the operator $\mathcal{M}$ satisfies

$$[\partial_{t_k} - B_k, \mathcal{M}] = 0 \quad (k = 1, 2, \ldots). \tag{3.3}$$

Also $\mathcal{M}$ is expressed as

$$\mathcal{M} = W\vartheta W^{-1} - \sum_{k=1,2,\ldots} n_k t_k W\Lambda_k W^{-1}.$$

Now we require that the similarity condition $\mathcal{M} \in \mathfrak{g}_{\geq 0}$ is satisfied. Note that it is equivalent to

$$\vartheta + \sum_{k=1,2,\ldots} n_k t_k B_k^c = W \vartheta W^{-1},$$

where B_k^c stands for the $\mathfrak{g}_{<0}$-component of $W \Lambda_k W^{-1}$. Then we have

$$\mathcal{M} = \vartheta - \sum_{k=1,2,\ldots} n_k t_k B_k.$$

We also assume that $t_k = 0$ for $k \geq 3$. Then the systems (3.2) and (3.3) are equivalent to

$$\begin{aligned} &[\partial_{t_1} - B_1, \partial_{t_2} - B_2] = 0, \\ &[\partial_{t_k} - B_k, \vartheta - t_1 B_1 - t_2 B_2] = 0 \quad (k = 1, 2). \end{aligned} \tag{3.4}$$

We regard the system (3.4) as a similarity reduction of the Drinfeld-Sokolov hierarchy of type $D_{2n+2}^{(1)}$.

The $\mathfrak{g}_{\geq 0}$-valued functions B_k $(k = 1, 2)$ are expressed in the form

$$B_k = U_k + \Lambda_k, \quad U_k = \sum_{i=0}^{2n+2} u_{k,i} \alpha_i^\vee + \sum_{i \in \mathcal{I}} x_{k,i} e_i + \sum_{i \in \mathcal{I}} y_{k,i} f_i.$$

In terms of the operators $U_k \in \mathfrak{g}_0$, this similarity reduction can be expressed as

$$\begin{aligned} &\partial_{t_1}(U_2) - \partial_{t_2}(U_1) + [U_2, U_1] = 0, \\ &[\Lambda_1, U_2] - [\Lambda_2, U_1] = 0, \\ &t_1 \partial_{t_1}(U_k) + t_2 \partial_{t_2}(U_k) + U_k = 0 \quad (k = 1, 2). \end{aligned}$$

In the following, we use the notation of a $\mathfrak{g}_{\geq 0}$-valued 1-form $\mathcal{B} = B_1 dt_1 + B_2 dt_2$ with respect to the coordinates $\boldsymbol{t} = (t_1, t_2)$. Then the similarity reduction (3.4) is expressed as

$$d_{\boldsymbol{t}} \mathcal{M} = [\mathcal{B}, \mathcal{M}], \quad d_{\boldsymbol{t}} \mathcal{B} = \mathcal{B} \wedge \mathcal{B}, \tag{3.5}$$

where $d_{\boldsymbol{t}}$ stands for an exterior differentation with respect to $\boldsymbol{t}$. Denoting by

$$\mathcal{M}_1 = -t_1 \Lambda_1 - t_2 \Lambda_2, \quad \mathcal{B}_1 = \Lambda_1 dt_1 + \Lambda_2 dt_2,$$

we can express the operators $\mathcal{M}$ and $\mathcal{B}$ in the form

$$\begin{aligned} \mathcal{M} &= \theta + \sum_{i \in \mathcal{I}} \xi_i e_i + \sum_{i \in \mathcal{I}} \psi_i f_i + \mathcal{M}_1, \\ \mathcal{B} &= \boldsymbol{u} + \sum_{i \in \mathcal{I}} \boldsymbol{x}_i e_i + \sum_{i \in \mathcal{I}} \boldsymbol{y}_i f_i + \mathcal{B}_1, \end{aligned}$$

where

$$\theta = \vartheta + \sum_{i=0}^{2n+2} \theta_i \alpha_i^\vee, \quad \boldsymbol{u} = \sum_{i=0}^{2n+2} \boldsymbol{u}_i \alpha_i^\vee.$$

The system (3.5) is expressed in terms of these variables as follows:

$$\begin{aligned} d_{\boldsymbol{t}}\theta_i &= \boldsymbol{x}_i\psi_i - \boldsymbol{y}_i\xi_i, \quad d_{\boldsymbol{t}}\theta_j = 0, \\ d_{\boldsymbol{t}}\xi_i &= (\boldsymbol{u}|\alpha_i^\vee)\xi_i - \boldsymbol{x}_i(\theta|\alpha_i^\vee), \\ d_{\boldsymbol{t}}\psi_i &= -(\boldsymbol{u}|\alpha_i^\vee)\psi_i + \boldsymbol{y}_i(\theta|\alpha_i^\vee), \end{aligned}$$

and

$$\begin{aligned} d_{\boldsymbol{t}}\boldsymbol{u}_i &= \boldsymbol{x}_i \wedge \boldsymbol{y}_i + \boldsymbol{y}_i \wedge \boldsymbol{x}_i, \quad d_{\boldsymbol{t}}\boldsymbol{u}_j = 0, \\ d_{\boldsymbol{t}}\boldsymbol{x}_i &= (\boldsymbol{u}|\alpha_i^\vee) \wedge \boldsymbol{x}_i, \quad d_{\boldsymbol{t}}\boldsymbol{y}_i = -(\boldsymbol{u}|\alpha_i^\vee) \wedge \boldsymbol{y}_i, \end{aligned}$$

for $i \in \mathcal{I}$ and $j \in \mathcal{J}$.

4 Coupled Painlevé VI system

In this section, we show that the system (1.1) with (1.2) is derived from the similarity reduction (3.5).

We introduce below a gauge transformation

$$\mathcal{M}^+ = \exp(\mathrm{ad}(\Gamma))\mathcal{M}, \quad d_{\boldsymbol{t}} - \mathcal{B}^+ = \exp(\mathrm{ad}(\Gamma))(d_{\boldsymbol{t}} - \mathcal{B}),$$

with $\Gamma \in \mathfrak{g}_0$ such that $\mathcal{M}^+$ and $\mathcal{B}^+$ should take values in $\mathfrak{b}_+$. Then the system (3.5) is transformed into

$$d_{\boldsymbol{t}}\mathcal{M}^+ = [\mathcal{B}^+, \mathcal{M}^+], \quad d_{\boldsymbol{t}}\mathcal{B}^+ = \mathcal{B}^+ \wedge \mathcal{B}^+.$$

It is equivalent to the system (1.1) with (1.2) under a certain transformation of variables. We recall that the operator $\mathcal{M}$ is expressed as

$$\mathcal{M} = \theta + \sum_{i\in\mathcal{I}} \xi_i e_i + \sum_{i\in\mathcal{I}} \psi_i f_i + \mathcal{M}_1,$$

where

$$\begin{aligned} \mathcal{M}_1 = &-t_1e_0 - t_2e_1 - \sum_{j\in\mathcal{J}'} t_1e_j - t_1e_{2n+1} - t_2e_{2n+2} - t_2e_{0,2} - t_1e_{1,2} \\ &- \sum_{j\in\mathcal{J}'} t_2(e_{j-1,j} + e_{j,j+1}) - t_2e_{2n,2n+1} - t_1e_{2n,2n+2} - \sum_{j\in\mathcal{J}'} t_1e_{j-1,j,j+1}. \end{aligned}$$

We first consider a gauge transformation

$$\mathcal{M}' = \exp(\mathrm{ad}(\Gamma_1))\mathcal{M}, \quad d_{\boldsymbol{t}} - \mathcal{B}' = \exp(\mathrm{ad}(\Gamma_1))(d_{\boldsymbol{t}} - \mathcal{B}),$$

with $\Gamma_1 = \sum_{i\in\mathcal{I}} \gamma_i e_i$ defined by

$$\gamma_2 = \frac{t_2}{t_1}, \quad \gamma_{2i+2} = \frac{t_1 + t_2\gamma_{2i}}{t_2 + t_1\gamma_{2i}} \quad (i = 1, \ldots, n-1).$$

Then we obtain

$$\begin{aligned}
\mathcal{M}'_1 &= \exp(\mathrm{ad}(\Gamma_1))(\mathcal{M}_1) \\
&= -t_1 e_0 - t_2 e_1 - \sum_{j\in\mathcal{J}'} t_1 e_j - t_2 e_{2n+1} - t_1 e_{2n+2} \\
&\quad - (t_1 - t_2\gamma_2)e_{1,2} - \sum_{j\in\mathcal{J}'} \{(t_2 + t_1\gamma_{j-1})e_{j-1,j} + (t_2 - t_1\gamma_{j+1})e_{j,j+1}\} \\
&\quad - (t_1 + t_2\gamma_{2n})e_{2n,2n+1} - (t_2 + t_1\gamma_{2n})e_{2n,2n+2}.
\end{aligned}$$

We next consider a gauge transformation

$$\mathcal{M}^* = \exp(\mathrm{ad}(\Gamma_2))\mathcal{M}', \quad d_{\boldsymbol{t}} - \mathcal{B}^* = \exp(\mathrm{ad}(\Gamma_2))(d_{\boldsymbol{t}} - \mathcal{B}'),$$

with $\Gamma_2 \in \mathfrak{h}$ such that

$$\begin{aligned}
\mathcal{M}^*_1 &= \exp(\mathrm{ad}(\Gamma_2))(\mathcal{M}'_1) \\
&= e_0 + b_1 e_1 + \sum_{j\in\mathcal{J}'} b_j e_j + b_{2n+1}e_{2n+1} + b_{2n+2}e_{2n+2} \\
&\quad + e_{1,2} + \sum_{j\in\mathcal{J}'} (e_{j-1,j} + e_{j,j+1}) + e_{2n,2n+1} + e_{2n,2n+2}.
\end{aligned}$$

Note that the coefficients b_j are algebraic functions in t_1 and t_2. Then We have

$$d_{\boldsymbol{t}}\mathcal{M}^* = [\mathcal{B}^*, \mathcal{M}^*], \quad d_{\boldsymbol{t}}\mathcal{B}^* = \mathcal{B}^* \wedge \mathcal{B}^*. \tag{4.1}$$

With the notation

$$\mathcal{B}^*_1 = \exp(\mathrm{ad}(\Gamma_2))\exp(\mathrm{ad}(\Gamma_1))(\mathcal{B}_1),$$

the operators $\mathcal{M}^*$ and $\mathcal{B}^*$ are expressed in the form

$$\begin{aligned}
\mathcal{M}^* &= \theta^* + \sum_{i\in\mathcal{I}} \xi^*_i e_i + \sum_{i\in\mathcal{I}} \psi^*_i f_i + \mathcal{M}^*_1, \\
\mathcal{B}^* &= \boldsymbol{u}^* + \sum_{i\in\mathcal{I}} \boldsymbol{x}^*_i e_i + \sum_{i\in\mathcal{I}} \boldsymbol{y}^*_i f_i + \mathcal{B}^*_1,
\end{aligned}$$

where

$$\theta^* = \vartheta + \sum_{i=0}^{2n+2} \theta_i^* \alpha_i^\vee, \quad \boldsymbol{u}^* = \sum_{i=0}^{2n+2} \boldsymbol{u}_i^* \alpha_i^\vee.$$

We finally consider a gauge transformation

$$\mathcal{M}^+ = \exp(\mathrm{ad}(\Gamma_3))\mathcal{M}^*, \quad d_{\boldsymbol{t}} - \mathcal{B}^+ = \exp(\mathrm{ad}(\Gamma_3))(d_{\boldsymbol{t}} - \mathcal{B}^*),$$

with $\Gamma_3 = \sum_{i\in\mathcal{I}} \eta_i f_i$ such that $\mathcal{M}^+, \mathcal{B}^+ \in \mathfrak{b}_+$, namely

$$\xi_i^* \eta_i^2 - (\theta^*|\alpha_i^\vee)\eta_i - \psi_i^* = 0 \quad (i \in \mathcal{I}), \tag{4.2}$$

and

$$d_{\boldsymbol{t}}\eta_i = \boldsymbol{x}_i^* \eta_i^2 - (\boldsymbol{u}^*|\alpha_i^\vee)\eta_i - \boldsymbol{y}_i^* \quad (i \in \mathcal{I}). \tag{4.3}$$

Here we have

Lemma 4.1. *Under the system* (4.1)*, the equation* (4.3) *follows from the equation* (4.2)*.*

Proof. The first equation of the system (4.1) can be expressed as

$$\begin{aligned} d_{\boldsymbol{t}}\theta_i^* &= \boldsymbol{x}_i^*\psi_i^* - \boldsymbol{y}_i^*\xi_i^*, \quad d_{\boldsymbol{t}}\theta_j^* = 0, \\ d_{\boldsymbol{t}}\xi_i^* &= (\boldsymbol{u}^*|\alpha_i^\vee)\xi_i^* - \boldsymbol{x}_i^*(\theta^*|\alpha_i^\vee), \\ d_{\boldsymbol{t}}\psi_i^* &= -(\boldsymbol{u}^*|\alpha_i^\vee)\psi_i^* + \boldsymbol{y}_i^*(\theta^*|\alpha_i^\vee), \end{aligned} \tag{4.4}$$

for $i \in \mathcal{I}$ and $j \in \mathcal{J}$. By using (4.4) and $(d_{\boldsymbol{t}}\theta^*|\alpha_i^\vee) = 2d_{\boldsymbol{t}}\theta_i^*$, we obtain

$$\begin{aligned} &d_{\boldsymbol{t}}\{\xi_i^*\eta_i^2 - (\theta^*|\alpha_i^\vee)\eta_i - \psi_i^*\} \\ &= \{2\xi_i^*\eta_i - (\theta^*|\alpha_i^\vee)\}\{d_{\boldsymbol{t}}\eta_i - \boldsymbol{x}_i^*\eta_i^2 + (\boldsymbol{u}^*|\alpha_i^\vee)\eta_i + \boldsymbol{y}_i^*\} \quad (i \in \mathcal{I}). \end{aligned}$$

It follows that the equation (4.2) implies (4.3) or

$$\eta_i = \frac{(\theta^*|\alpha_i^\vee)}{2\xi_i^*} \quad (i \in \mathcal{I}). \tag{4.5}$$

Hence it is enough to verify that the equation (4.3) follows from (4.5). Together with (4.4), the equation (4.5) implies

$$\begin{aligned} d_{\boldsymbol{t}}\eta_i &= \frac{(d_{\boldsymbol{t}}\theta^*|\alpha_i^\vee)\xi_i^* - (\theta^*|\alpha_i^\vee)d_{\boldsymbol{t}}\xi_i^*}{2(\xi_i^*)^2} \\ &= \boldsymbol{x}_i^*\eta_i^2 - (\boldsymbol{u}^*|\alpha_i^\vee)\eta_i - \boldsymbol{y}_i^* + \frac{\boldsymbol{x}_i^*\{4\xi_i^*\psi_i^* + (\theta^*|\alpha_i^\vee)^2\}}{4(\xi_i^*)^2} \quad (i \in \mathcal{I}). \end{aligned} \tag{4.6}$$

On the other hand, we obtain

$$4\xi_i^*\psi_i^* + (\theta^*|\alpha_i^\vee)^2 = 0 \quad (i \in \mathcal{I}), \tag{4.7}$$

by substituting (4.5) into (4.2). Combining (4.6) and (4.7), we obtain the equation (4.3). □

Thanks to Lemma 4.1, the gauge parameters η_i $(i \in \mathcal{I})$ are determined by the equation (4.2). Hence we obtain the system on $\mathfrak{b}_+$

$$d_{\boldsymbol{t}}\mathcal{M}^+ = [\mathcal{B}^+, \mathcal{M}^+], \quad d_{\boldsymbol{t}}\mathcal{B}^+ = \mathcal{B}^+ \wedge \mathcal{B}^+, \tag{4.8}$$

with dependent variables

$$\lambda_i = \eta_i - \sum_{j=1}^{i-1} b_{2j+1}, \quad \mu_i = \varphi_i^* \quad (i \in \mathcal{I}).$$

The operator $\mathcal{M}^+$ is expressed in the form

$$\begin{aligned}\mathcal{M}^+ = \kappa + \sum_{i\in\mathcal{I}} \mu_i e_i + e_0 + (c_1 - \lambda_2)e_1 + \sum_{j\in\mathcal{J}'}(\lambda_{j-1} - \lambda_{j+1})e_j \\ + (\lambda_{2n} - c_{2n+1})e_{2n+1} + (\lambda_{2n} - c_{2n+2})e_{2n+2} \\ + e_{1,2} + \sum_{j\in\mathcal{J}'}(e_{j-1,j} + e_{j,j+1}) + e_{2n,2n+1} + e_{2n,2n+2},\end{aligned}$$

where $\kappa \in \mathfrak{h}$ and

$$c_1 = b_1, \quad c_i = -\sum_{j=1}^{n-1} b_{2j+1} - b_i \quad (i = 2n+1, 2n+2).$$

Note that $d_{\boldsymbol{t}}\kappa = 0$. We also remark that c_1, c_{2n+1} and c_{2n+2} are algebraic functions in t_1 and t_2.

Let

$$s_1 = \frac{c_{2n+2} - c_1}{2n+2}, \quad s_2 = \frac{c_{2n+2} - c_{2n+1}}{2n+2}.$$

We now regard the system (4.8) as a system of ordinary differential equations

$$\left[s(s-1)\frac{d}{ds} - B, \mathcal{M}^+\right] = 0, \tag{4.9}$$

with respect to the independent variable $s = s_1$ by setting $s_2 = 1$. The explicit formula of the $\mathfrak{b}_+$-valued operator B is given below. We also set

$$q_i = \frac{c_{2n+2} - \lambda_{2i}}{2n+2}, \quad p_i = -\mu_{2i}, \quad \alpha_j = \frac{(\kappa|\alpha_j^\vee)}{2n+2},$$

for $i = 1, \ldots, n$ and $j = 0, \ldots, 2n+2$. Then we obtain

Theorem 4.2. *The system* (4.9) *is equivalent to the system* (1.1) *with* (1.2).

The operator $\mathcal{M}^+$ is described as

$$\mathcal{M}^+ = \kappa + \sum_{i=0}^{2n+2} (2n+2)\varphi_i e_i + \sum_{i=1}^{2n} e_{i,i+1} + e_{2n,2n+2},$$

We recall that

$$\begin{aligned}
&\varphi_0 = \frac{1}{2n+2}, \quad \varphi_1 = q_1 - s, \quad \varphi_{2i+1} = q_{i+1} - q_i \quad (i = 1, \ldots, n-1),\\
&\varphi_{2j} = -\frac{p_j}{2n+2} \quad (j = 1, \ldots, n), \quad \varphi_{2n+1} = 1 - q_n, \quad \varphi_{2n+2} = -q_n.
\end{aligned}$$

The operator B is described as

$$B = u + \sum_{i=0}^{2n+2} x_i e_i + y_1 e_{0,2} + \sum_{i=2}^{2n} y_i e_{i,i+1} + y_{2n+1} e_{2n,2n+2} + \sum_{j \in \mathcal{J}'} y_1 e_{j-1,j,j+1},$$

where

$$\begin{aligned}
&x_0 = -\frac{q_1 - s}{2n+2}, \quad x_1 = 1, \quad x_{2i+1} = s(s-1) - (q_i - s)(q_{i+1} - s),\\
&x_{2n+1} = -(s-1)q_n, \quad x_{2n+2} = -s(q_n - 1),\\
&y_1 = -\frac{1}{(2n+2)^2}, \quad y_{2i} = -\frac{q_{i+1} - s}{2n+2}, \quad y_{2i+1} = \frac{q_i - s}{2n+2},\\
&y_{2n} = \frac{s-1}{2n+2}, \quad y_{2n+1} = \frac{s}{2n+2},
\end{aligned}$$

for $i = 1, \ldots, n-1$ and

$$(2n+2)x_{2i} = \sum_{j=1}^{i-1} 2\{(q_j - s)p_j + \alpha_{2j}\} + (q_i - s)p_i + \alpha_{2i} + \alpha_0 + \sum_{j=1}^{i-1} \alpha_{2j+1},$$

for $i=1,\dots,n$. Here $u=\sum_{i=0}^{2n+2}u_i\alpha_i^\vee$ satisfies

$$
\begin{aligned}
(u|\alpha_0^\vee) &= -\alpha_0(q_1-s),\\
(u|\alpha_1^\vee) &= -\alpha_0(q_1+s-1)-\sum_{j=1}^{n}2q_j\{(q_j-1)p_j+\alpha_{2j}\}\\
&\quad -(2\alpha_2+\beta_{1,3})(s-1)-\beta_{1,4}s,\\
(u|\alpha_{2i+1}^\vee) &= -\left\{\sum_{j=1}^{i}2(q_j-s)p_j+\beta_{i,1}+2\alpha_{2i}\right\}(q_i+q_{i+1}-1)-\beta_{i+1,4}s\\
&\quad -\sum_{j=i+1}^{n}2q_j\{(q_j-1)p_j+\alpha_{2j}\}-(2\alpha_{2i+2}+\beta_{i+1,3})(s-1),\\
(u|\alpha_{2n+1}^\vee) &= -\left\{\sum_{j=1}^{n}2(q_j-s)p_j+\beta_{n,1}+2\alpha_{2n}\right\}q_n-\alpha_{2n+1}s,\\
(u|\alpha_{2n+2}^\vee) &= -\left\{\sum_{j=1}^{n}2(q_j-s)p_j+\beta_{n,1}+2\alpha_{2n}\right\}(q_n-1)-\alpha_{2n+2}(s-1),
\end{aligned}
$$

for $i=1,\dots,n-1$ and

$$
\begin{aligned}
(u|\alpha_{2i}^\vee) &= \left\{\sum_{j=1}^{i-1}2(q_j-s)p_j+(q_i-s)p_i+\beta_{i,1}+2\alpha_{2i}\right\}(2q_i-1)\\
&\quad +q_i\{(q_i-1)p_i+\alpha_{2i}\}+\sum_{j=i+1}^{n}2q_j\{(q_j-1)p_j+\alpha_{2j}\}\\
&\quad +(2\alpha_{2i}+\beta_{i+1,3})(s-1)+\beta_{i+1,4}s,
\end{aligned}
$$

for $i=1,\dots,n$, where

$$
\begin{aligned}
&\beta_{i,0}=\alpha_1+\sum_{j=1}^{i-1}\alpha_{2j+1}, \quad \beta_{i,1}=\alpha_0+\sum_{j=1}^{i-1}2\alpha_{2j}+\sum_{j=1}^{i-1}\alpha_{2j+1},\\
&\beta_{i,3}=\sum_{j=i}^{n-1}\alpha_{2j+1}+\sum_{j=i+1}^{n}2\alpha_{2j}+\alpha_{2n+1}, \quad \beta_{i,4}=\sum_{j=i}^{n-1}\alpha_{2j+1}+\alpha_{2n+2}.
\end{aligned}
$$

Remark 4.3. *The system* (1.1) *with* (1.2) *is derived from a Lax pair associated with the loop algebra* $\mathfrak{so}(4n+4)[z,z^{-1}]$*; see Appendix B.*

5 Affine Weyl group symmetry

In this section, we discuss a derivation of the group of symmetries (1.3) following the manner in [NY2].

Recall that the affine Weyl group of type $D^{(1)}_{2n+2}$ is generated by the transformations r_i $(i = 0, \ldots, 2n+2)$ with the fundamental relations

$$\begin{aligned} &r_i^2 = 1 && (i = 0, \ldots, 2n+2), \\ &(r_i r_j)^{2-a_{ij}} = 0 && (i, j = 0, \ldots, 2n+2; i \neq j). \end{aligned}$$

acting on the simple roots as

$$r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i \quad (i, j = 0, \ldots, 2n+2),$$

where

$$\begin{aligned} &a_{ii} = 2 && (i = 0, \ldots, 2n+2), \\ &a_{02} = a_{ii+1} = a_{2n2n+2} = -1 && (i = 1, \ldots, 2n), \\ &a_{ij} = 0 && (\text{otherwise}). \end{aligned}$$

Let $X(0) \in G_{<0}G_{\geq 0}$. We consider a $G_{<0}G_{\geq 0}$-valued function

$$X = X(t_1, t_2, \ldots) = \exp\left(\sum_{k=1,2,\ldots} t_k \Lambda_k\right) X(0).$$

Then we have a system of partial differential equations

$$X \partial_k X^{-1} = \partial_k - \Lambda_k \quad (k = 1, 2, \ldots),$$

defined through the adjoint action of $G_{<0}G_{\geq 0}$ on $\widehat{\mathfrak{g}}_{<0} \oplus \mathfrak{g}_{\geq 0}$. Via a decomposition

$$X = W^{-1}Z, \quad W \in G_{<0}, \quad Z \in G_{\geq 0},$$

we obtain the system (3.1).

In the previous section, we have considered the gauge transformation

$$\mathcal{M}^+ = \exp(\mathrm{ad}(\Gamma))\mathcal{M}, \quad d_{\boldsymbol{t}} - \mathcal{B}^+ = \exp(\mathrm{ad}(\Gamma))(d_{\boldsymbol{t}} - \mathcal{B}), \quad \Gamma \in \mathfrak{g}_0,$$

for the derivation of the system (1.1). Note that it arises from

$$X = (W^+)^{-1}Z^+, \quad W^+ = \exp(\Gamma)W, \quad Z^+ = \exp(\Gamma)Z.$$

Consider transformations

$$r_i(X) = X \exp(-e_i)\exp(f_i)\exp(-e_i) \quad (i = 0, \ldots, 2n+2).$$

Under the similarity condition $\mathcal{M}^+ \in \mathfrak{b}_+$, their action on W^+ is given by

$$r_i(W^+) = G_i W^+ \quad (i = 0, \ldots, 2n+2),$$

where

$$G_i = \exp\left(\frac{\alpha_i}{\varphi_i} f_i\right), \quad \alpha_i = \frac{(\alpha_i^\vee|\mathcal{M}^+)}{2n+2}, \quad \varphi_i = \frac{(f_i|\mathcal{M}^+)}{2n+2}.$$

It follows that

$$r_i(\mathcal{M}^+) = G_i \mathcal{M}^+ G_i^{-1}, \quad d_{\boldsymbol{t}} - r_i(\mathcal{B}^+) = G_i(d_{\boldsymbol{t}} - \mathcal{B}^+)G_i^{-1},$$

for $i = 0, \ldots, 2n+2$. Then each $r_i(\mathcal{M}^+)$ and $r_i(\mathcal{B}^+)$ are $\mathfrak{b}_+$-valued and satisfy the system (4.8). Note that the complex parameters α_i $(i = 0, \ldots, 2n+2)$ can be regarded as the simple roots for $\mathfrak{g}(D^{(1)}_{2n+2})$.

We define a Poisson structure for the operator $\mathcal{M}^+$ by

$$\{\varphi_i, \varphi_j\} = \frac{([f_j, f_i]|\mathcal{M}^+)}{2n+2} \quad (i, j = 0, \ldots, 2n+2).$$

It is equivalent to

$$\{p_i, q_j\} = \delta_{i,j}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0 \quad (i, j = 1, \ldots, n).$$

Hence p_i, q_i $(i = 1, \ldots, n)$ give a canonical coordinate system associated with the Poisson structure for $\mathcal{M}^+$. Then the action of the transformations r_i $(i = 0, \ldots, 2n+2)$ on the coefficients of $\mathcal{M}^+$ is equivalent to (1.3).

Remark 5.1 ([S])**.** *Let*

$$\sigma_1(i) = (0,1)(2n+1, 2n+2)i, \quad \sigma_2(i) = 2n+2-i \quad (i = 0, \ldots, 2n+2),$$

where $(0,1)$ *and* $(2n+1, 2n+2)$ *stand for the adjacent transpositions. Then the system* (1.1) *with* (1.2) *is invariant under the action of transformations* π_1 *and* π_2 *defined by*

$$\pi_1(\alpha_i) = \alpha_{\sigma_1(i)}, \quad \pi_1(q_i) = \frac{s(q_i - 1)}{q_i - s}, \quad \pi_1(p_i) = \frac{(q_i - s)\{p_i(q_i - s) + \alpha_{2i}\}}{s(1-s)},$$

and

$$\pi_2(\alpha_i) = \alpha_{\sigma_2(i)}, \quad \pi_2(q_i) = \frac{s}{q_i}, \quad \pi_2(p_i) = -\frac{q_i(q_i p_i + \alpha_{2i})}{s},$$

for $i = 0, \ldots, 2n+2$*. These transformations generate a group of Dynkin diagram automorphisms of type* $D^{(1)}_{2n+2}$*. In fact, they satisfy the fundamental relations*

$$\begin{aligned} &\pi_i^2 = 1 && (i = 1, 2), \\ &(\pi_1\pi_2)^3 = 1, && \\ &\pi_i r_j = r_{\sigma_i(j)}\pi_i && (i = 1, 2; j = 0, \ldots, 2n+2). \end{aligned}$$

A Heisenberg subalgebra

We first introduce the simple Lie algebra $\mathfrak{so}(4n+4)$ and its loop algebra. Denoting matrix units by $E_{i,j} = (\delta_{i,k}\delta_{j,l})_{k,l=1}^{4n+4}$, we set

$$J = \sum_{i=1}^{4n+4} E_{i,4n+5-i}.$$

Then the algebra $\mathfrak{so}(4n+4)$ is defined by

$$\mathfrak{so}(4n+4) = \left\{ X \in \mathrm{Mat}(4n+4;\mathbb{C}) \;\middle|\; JX + {}^{t}XJ = 0 \right\}.$$

Also let E_j, F_j, H_j $(j = 0, \dots, 2n+2)$ be the Chevalley generators for the loop algebra $\mathfrak{so}(4n+4)[z, z^{-1}]$ defined by

$$\begin{aligned}
&E_0 = zX_{4n+3,1}, \quad E_i = X_{i,i+1}, \quad E_{2n+2} = X_{2n+1,2n+3}, \\
&F_0 = \frac{1}{z}X_{1,4n+3}, \quad F_i = X_{i+1,i}, \quad F_{2n+2} = X_{2n+3,2n+1}, \\
&H_0 = -X_{1,1} - X_{2,2}, \quad H_i = X_{i,i} - X_{i+1,i+1}, \\
&H_{2n+2} = X_{2n+1,2n+1} + X_{2n+2,2n+2},
\end{aligned}$$

for $i = 1, \dots, 2n+1$, where $X_{i,j} = E_{i,j} - E_{4n+5-j,4n+5-i}$. Note that

$$H_0 + H_1 + \sum_{i=2}^{2n} 2H_i + H_{2n+1} + H_{2n+2} = 0.$$

Under a specialization $K = 0$, we can identify this loop algebra with the affine Lie algebra $\mathfrak{g}(D_{2n+2}^{(1)})$. Note that the scaling element d corresponds to the differential operator $z\partial_z$. We also remark that

$$[X, Y] = XY - YX, \quad (X|Y) = \frac{1}{2}\mathrm{tr}XY.$$

In a similar manner as [DF], we formulate the Heisenberg subalgebra of type $(1,1,0,1,0,\dots,1,0,1,1)$ in a flamework of $\mathfrak{so}(4n+4)[z, z^{-1}]$. Let $\Lambda_{1,i}$ $(i = 1, 2)$ be matricies defined by

$$\begin{aligned}
\Lambda_{1,1} &= E_0 + [E_1, E_2] + \sum_{j\in\mathcal{J}'} (E_j + [E_{j-1}, [E_j, E_{j+1}]]) + E_{2n+1} + [E_{2n}, E_{2n+2}], \\
\Lambda_{1,2} &= E_1 + [E_0, E_2] + \sum_{j\in\mathcal{J}'} ([E_{j-1}, E_j] + [E_j, E_{j+1}]) + E_{2n+2} + [E_{2n}, E_{2n+1}].
\end{aligned}$$

Note that $[\Lambda_{1,1}, \Lambda_{1,2}] = 0$. We also set

$$\Lambda_{(2n+2)k+l,i} = z^k (\Lambda_{1,i})^l \quad (i = 1, 2; k \in \mathbb{Z}; l = 1, 3, \ldots, 2n+1).$$

Then we have a maximal nilpotent subalgebra $\bigoplus_{k\in\mathbb{Z}} (\mathbb{C}\Lambda_{2k-1,1} \oplus \mathbb{C}\Lambda_{2k-1,2})$ of $\mathfrak{so}(4n+4)[z, z^{-1}]$. It can be identified with the Heisenberg subalgebra $\mathfrak{s}$ given in Section 2 under the specialization $K = 0$.

Remark A.1. *The isomorphism classes of the Heisenberg subalgebras are in one-to-one correspondence with the conjugacy classes of the finite Weyl group* [KP]*. In the notation of* [C]*, the Heisenberg subalgebra $\mathfrak{s}$ introduced above corresponds to the regular primitive conjugacy class $D_{2n+2}(a_n)$ of the Weyl group of type D_{2n+2}.*

B Lax pair

It is known that P_{VI} is derived from the Lax pair associated with the loop algebra $\mathfrak{so}(8)[z, z^{-1}]$ [NY3]. In this section, we propose a Lax pair for the system (1.1) with (1.2) in a flamework of $\mathfrak{so}(4n+4)[z, z^{-1}]$.

In the previous section, we have derived the system (4.9). It can be identified with the system on $\mathfrak{so}(4n+4)[z, z^{-1}]$

$$\left[s(s-1)\frac{d}{ds} - B, z\frac{z}{dz} + M\right] = 0, \tag{B.1}$$

where

$$\begin{aligned}
M &= \sum_{i=0}^{2n+2} \varepsilon_i H_i + \sum_{i=0}^{2n+2} \varphi_i E_i + \sum_{i=1}^{2n} \frac{[E_i, E_{i+1}]}{2n+2} + \frac{[E_{2n}, E_{2n+2}]}{2n+2}, \\
B &= \sum_{i=0}^{2n+2} u_i H_i + \sum_{i=0}^{2n+2} x_i E_i + y_1 [E_0, E_2] + \sum_{i=2}^{2n} y_i [E_i, E_{i+1}] \\
&\quad + y_{2n+1} [E_{2n}, E_{2n+2}] + \sum_{j\in\mathcal{J}'} y_1 [E_{j-1}, [E_j, E_{j+1}]],
\end{aligned}$$

under the specialization $K = 0$. Here ε_i $(i = 0, \ldots, 2n+2)$ are complex parameters such as

$$\begin{aligned}
&\alpha_0 = 1 + 2\varepsilon_0 - \varepsilon_2, \quad \alpha_1 = 2\varepsilon_1 - \varepsilon_2, \quad \alpha_2 = -\varepsilon_0 - \varepsilon_1 + 2\varepsilon_2 - \varepsilon_3, \\
&\alpha_i = -\varepsilon_{i-1} + 2\varepsilon_i - \varepsilon_{i+1} \quad (i = 3, \ldots, 2n-1), \\
&\alpha_{2n} = -\varepsilon_{2n-1} + 2\varepsilon_{2n} - \varepsilon_{2n+1} - \varepsilon_{2n+2}, \\
&\alpha_{2n+1} = -\varepsilon_{2n} + 2\varepsilon_{2n+1}, \quad \alpha_{2n+2} = -\varepsilon_{2n} + 2\varepsilon_{2n+2}.
\end{aligned}$$

Consider a system of linear differential equations

$$s(s-1)\frac{d\boldsymbol{w}}{ds} = B\boldsymbol{w}, \quad z\frac{d\boldsymbol{w}}{dz} + M\boldsymbol{w} = 0, \tag{B.2}$$

for a vector of unknown functions $\boldsymbol{w} = {}^t(w_1, \ldots, w_{4n+4})$. Then the system (B.1) can be regarded as the compatibility condition of (B.2). In this flamework, the group of symmetries (1.3) arise from gauge transformations

$$r_i(\boldsymbol{w}) = \left(1 + \frac{\alpha_i}{\varphi_i}F_i\right)\boldsymbol{w} \quad (i = 0, \ldots, 2n+2).$$

Note that the Lax pair (B.2) of the case $n = 1$ is equivalent to one of [NY3].

The Lax pair (B.2) arises from the Drinfeld-Sokolov hierarchy as follows. Under the system (3.1), we consider a $G_{<0}G_{\geq 0}$-function $\Psi = \Psi(t_1, t_2, \ldots)$ defined by

$$\Psi = W \exp\left(\sum_{k=1,2,\ldots} t_k \Lambda_k\right).$$

Then we obtain

$$\Psi \partial_{t_k} \Psi^{-1} = \partial_{t_k} - B_k \quad (k = 1, 2, \ldots), \quad \Psi\vartheta\Psi^{-1} = \mathcal{M}. \tag{B.3}$$

Note that the system (3.2) can be regarded as the compatibility condition of (B.3). In the following, we use a conventional form of (B.3)

$$\partial_{t_k}(\Psi) = B_k\Psi \quad (k = 1, 2, \ldots), \quad \vartheta(\Psi) = (\vartheta - \mathcal{M})\Psi.$$

It is equivalent to

$$\partial_{t_k}(\Psi) = B_k\Psi \quad (k = 1, 2), \quad \vartheta(\Psi) = (t_1B_1 + t_2B_2)\Psi, \tag{B.4}$$

under the specialization $\mathcal{M} \in \mathfrak{g}_{\geq 0}$ and $t_k = 0$ $(k \geq 3)$. Via a gauge transformation $\Psi^+ = \exp(\Gamma)\Psi$, the system (B.4) is transformed into

$$\partial_{s_k}(\Psi^+) = B_k^+\Psi^+ \quad (k = 1, 2), \quad \vartheta(\Psi^+) = (\vartheta - \mathcal{M}^+)\Psi^+, \tag{B.5}$$

where B_k^+ $(k = 1, 2)$ are defined by $\mathcal{B}^+ = B_1^+ds_1 + B_2^+ds_2$ and

$$s_1 = \frac{c_{2n+2} - c_1}{2n+2}, \quad s_2 = \frac{c_{2n+2} - c_{2n+1}}{2n+2}.$$

The system (B.5) can be identified with (B.2) under the specialization $s_2 = 1$ and $K = 0$.

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